*Electronic Journal of Differential Equations*, Vol. 2004(2004), No. 144, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu (login: ftp)

# QUASILINEAR ELLIPTIC SYSTEMS IN DIVERGENCE FORM WITH WEAK MONOTONICITY AND NONLINEAR PHYSICAL DATA

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Abstract. We study the quasilinear elliptic system

$$-\operatorname{div}\sigma(x,u,Du)=v(x)+f(x,u)+\operatorname{div}g(x,u)$$

on a bounded domain of $\mathbb{R}^n$ with homogeneous Dirichlet boundary conditions. This system corresponds to a diffusion problem with a source $v$ in a moving and dissolving substance, where the motion is described by $g$ and the dissolution by $f$. We prove existence of a weak solution of this system under classical regularity, growth, and coercivity conditions for $\sigma$, but with only very mild monotonicity assumptions.

## 1. Introduction

Let $\Omega$ denote a bounded open domain in $\mathbb{R}^n$. Let $\mathbb{M}^{m\times n}$ denote the set of real $m$ by $n$ matrices equipped with the usual inner product $A:B=A_{ij}B_{ij}$. In [12] the following quasilinear elliptic system was cosidered:

$$\begin{gathered}-\operatorname{div}\,\sigma(x,u,Du)=v(x)\quad\text{on }\Omega\\ u=0\quad\text{on }\partial\Omega\,,\end{gathered}\tag{1.1}$$

where $v$ belongs to the dual space $W^{-1,p'}(\Omega;\mathbb{R}^m)$ of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ and $\sigma$ satisfies the following conditions for some $p\in(1,\infty)$:

(E0) (Continuity) $\sigma:\Omega\times\mathbb{R}^m\times\mathbb{M}^{m\times n}\to\mathbb{M}^{m\times n}$ is a Carathéodory function, i.e. $x\mapsto\sigma(x,u,F)$ is measurable for every $(u,F)\in\mathbb{R}^m\times\mathbb{M}^{m\times n}$ and $(u,F)\mapsto\sigma(x,u,F)$ is continuous for almost every $x\in\Omega$.

(E1) (Growth and coercivity) There exist $c_1\geqslant 0$, $c_2>0$, $\lambda_1\in L^{p'}(\Omega)$, $\lambda_2\in L^1(\Omega)$, $0<\alpha<p$, $\lambda_3\in L^{(p/\alpha)'}(\Omega)$ and $0<\beta\leqslant\frac{n}{n-p}(p-1)$ such that

$$\begin{gathered}|\sigma(x,u,F)|\leqslant\lambda_1(x)+c_1(|u|^\beta+|F|^{p-1})\\ \sigma(x,u,F):F\geqslant-\lambda_2(x)-\lambda_3(x)|u|^\alpha+c_2|F|^p\end{gathered}$$

(E2) (Monotonicity) $\sigma$ satisfies one of the following conditions:

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2000 *Mathematics Subject Classification.* 35J60.
*Key words and phrases.* Young measure; noninear elliptic systems.

Submitted August 16, 2004. Published December 7, 2004.

(a) For all $x \in \Omega$ and all $u \in \mathbb{R}^m$, the map $F \mapsto \sigma(x,u,F)$ is a $C^1$-function and is monotone, i.e.
$$(\sigma(x,u,F) - \sigma(x,u,G)) : (F - G) \geqslant 0$$
for all $x \in \Omega$, $u \in \mathbb{R}^m$ and $F, G \in \mathbb{M}^{m\times n}$.

(b) There exists a function $W : \Omega\times\mathbb{R}^m\times\mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x,u,F) = \frac{\partial W}{\partial F}(x,u,F)$, and $F \mapsto W(x,u,F)$ is convex and $C^1$.

(c) For all $x \in \Omega$ and all $u \in \mathbb{R}^m$, the map $F \mapsto \sigma(x,u,F)$ is strictly monotone, i.e. $\sigma(x,u,\cdot)$ is monotone and $(\sigma(x,u,F) - \sigma(x,u,G)) : (F - G) = 0$ implies $F = G$.

(d) $\sigma(x,u,F)$ is strictly $p$-quasimonotone in $F$ i.e.
$$\int_{\mathbb{M}^{m\times n}} (\sigma(x,u,\lambda) - \sigma(x,u,\bar\lambda)) : (\lambda - \bar\lambda) d\nu(\lambda) > 0$$
for all $x \in \Omega$, all $u \in \mathbb{R}^m$ and all homogeneous $W^{1,p}$ gradient Young measures $\nu$ with center of mass $\bar\lambda = \langle \nu, \mathrm{id}\rangle$ which are not a single Dirac mass.

Condition (E0) ensures that $\sigma(x,u(x),U(x))$ is measurable on $\Omega$ for measurable functions $u : \Omega \to \mathbb{R}^m$ and $U : \Omega \to \mathbb{M}^{m\times n}$; see e.g. [20, Appendix "Measurable funtions" (12), page 1013].

Condition (E1) states standard growth and coercivity conditions. The main point is that we do not require strict monotonicity of a typical Leray-Lions operator [15] or monotonicity in the variables $(u,F)$ in (E2) as it is usually assumed in previous works. Thus, the classical monotone operator methods [6, 15, 4, 21, 20] developed by Višik, Minty, Browder, Brézis, Lions and others do not apply in general for functions satisfying only (E0)–(E2).

For example, the assumption (E2) allows to take a potential $W(x,u,F)$, which is only convex but not strictly convex in $F$, and to consider the corresponding elliptic problem (QES) with $\sigma(x,u,F) = \frac{\partial W}{\partial F}(x,u,F)$. Even such a very simple situation cannot be treated by conventional methods. The problem is that the gradients of approximating solutions do not need to converge pointwise where $W$ is not strictly convex. The idea is now, that in a point where $W$ is not strictly convex, it is locally affine, and therefore, passage to the limit should locally still be possible. Technically, this can indeed be achieved by considering the Young measure generated by the sequence of gradients of approximating solutions.

The assumption (d) in (E2) is motivated by the study of nonlinear elastostatics by Ball. For non-hyperelastic materials the static equation is not given by a potential map. Subsequently quasimonotone systems have been studied by Zhang and Chabrowski [7] who investigated the existence of solutions for perturbed systems. However, a slightly different notion of quasimonotonicity is used in the mentioned papers. The regularity problems for such systems were studied by Fuchs [10]. A simple example of a strictly $p$-quasimonotone function is the following:

**Example 1.1.** Assume that $\eta : \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ satisfies the growth condition
$$|\eta(F)| \leqslant C\,|F|^{p-1}$$
with $p > 1$ and the structure condition
$$\int_\Omega (\eta(F + \nabla\phi) - \eta(F)) : \nabla\phi\, dx \geqslant c \int_\Omega |\nabla\phi|^r dx$$

for constants $c > 0$, $r > 0$, and for all $\phi \in C_0^\infty(\Omega; \mathbb{R}^m)$ and all $F \in \mathbb{M}^{m\times n}$. Then $\eta$ is strictly $p$-quasimonotone. This follows easily from the definition if one uses that for every $W^{1,p}$ gradient Young measure $\nu$ there exists a sequence $(Dv_k)$ generating $\nu$ for which $(|Dv_k|^p)$ is equiintegrable [13].

An example of an operator which satisfies all conditions (E0)–(E2) is the $p$-Laplace operator $\Delta_p$ (which in fact is even uniformly monotone).

**Example 1.2.** The function $\sigma(x,u,Du)=|Du|^{p-2}Du$ satisfies (E0)–(E2). Conditions (E0), (E1) are obvious and (a), (c) and (d) in (E2) follow by direct calculations. For (b), we may choose $W(x,u,F) = \frac{1}{p}|F|^p$. Note that we have then

$$\operatorname{div}\ |Du|^{p-2}Du = \Delta_p u.$$

In [12], Young measures are used to prove the needed compactness of the approximating solutions obtained through a Galerkin scheme. With this method, it is shown in [11] that the Dirichlet problem (QES) has a weak solution $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ under the mild assumptions (E0)–(E2).

**Definition 1.3.** We say that $u : \Omega \to \mathbb{R}^m$ is a weak solution of

$$\begin{gathered} -\operatorname{div}\big(a(x,u,Du)\big) + b(x,u,Du) = v(x) \quad \text{on } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered}$$

with $v \in W^{-1,p'}(\Omega;\mathbb{R}^m)$ if:

(i) $u$ belongs to $W_0^{1,1}(\Omega;\mathbb{R}^m)$
(ii) $a(\cdot,u(\cdot),Du(\cdot))$ belongs to $L^1(\Omega;\mathbb{M}^{m\times n})$ and $b(\cdot,u(\cdot),Du(\cdot))$ to $L^1(\Omega;\mathbb{R}^m)$,
(iii) the equality

$$\int_\Omega a(x,u(x),Du(x)) : D\varphi(x)dx + \int_\Omega b(x,u(x),Du(x))\cdot\varphi(x)dx = \langle v,\varphi\rangle$$

holds for every function $\varphi \in C_c^\infty(\Omega;\mathbb{R}^m)$.

Here, $\langle\cdot,\cdot\rangle$ denotes the dual pairing of $W^{-1,p'}$ and $W^{1,p}$.

**Remark 1.4.** In Definition 1.3 the boundary condition $u = 0$ on $\partial\Omega$ is interpreted in the sense of (i).

The purpose of this article is, motivated by physics or geometry, to generalize the right hand side of (1.1) and to prove the existence of a weak solution, again under the weak assumptions (E0)–(E2). In this sense, for a function $u : \Omega \to \mathbb{R}^m$, we consider the quasilinear elliptic system, $(\mathrm{QES})_{f,g}$,

$$\begin{gathered} -\operatorname{div}\ \sigma(x,u,Du) = v(x) + f(x,u) + \operatorname{div}\ g(x,u) \quad \text{on } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,, \end{gathered} \tag{1.2}$$

which is a Dirichlet problem. Here, $\sigma$ satisfies (E0)–(E2) for some $p \in (1,\infty)$, $v \in W^{-1,p'}(\Omega;\mathbb{R}^m)$ and $f$ and $g$ satisfy the following continuity and growth conditions:

(F0) (Continuity) $f : \Omega\times\mathbb{R}^m \to \mathbb{R}^m$ is a Carathéodory function, i.e. $x \mapsto f(x,u)$ is measurable for every $u \in \mathbb{R}^m$ and $u \mapsto f(x,u)$ is continuous for almost every $x \in \Omega$.
(F1) (Growth) There exist $0 < \gamma < p-1$, $b_1 \in L^{p'}(\Omega)$ and $b_2 \in L^{\frac{n}{p}}(\Omega)$ such that

$$|f(x,u)| \leqslant b_1(x) + b_2(x)|u|^\gamma.$$

(G0) (Continuity) $g : \Omega\times\mathbb{R}^m \to \mathbb{M}^{m\times n}$ is a Carathéodory function.

(G1) (Growth) There exist $0 < \eta < p-1$, $b_4 \in L^{p'}(\Omega)$ and $b_5 \in L^{\frac{n}{p-1}}(\Omega)$ such that
$$|g(x,u)| \leqslant b_4(x) + b_5(x)|u|^\eta.$$

Conditions (F0) and (G0) ensure that $f(x,u(x))$ and $g(x,u(x))$ are measurable on $\Omega$ for any measurable function $u : \Omega \to \mathbb{R}^m$. (F1) and (G1) state standard growth conditions. In particular, if $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ then $f(\cdot,u(\cdot)) \cdot u(\cdot)$ and $g(\cdot,u(\cdot)) : Du(\cdot)$ belong to $L^1(\Omega)$.

The notation $(\text{QES})_{f,g}$ for (1.2) should help the reader and lighten the text. So that the subscripts $f$ and $g$ show the nature of the right hand side. In particular, when $g = 0$, the system will be denoted by $(\text{QES})_f$. Note that the system $(\text{QES})_{f,g}$ is more general than the system $(\text{QES})_f$. Indeed, the term $\operatorname{div} g$ cannot be absorbed in $\operatorname{div}\sigma$ or in $f$ since no condition on the derivatives of $g$ or on the monotonicity of $f$ and $g$ is imposed. Adapting the methods used in [11], we will prove the existence of a weak solution for the system (1.2):

**Theorem 1.5.** *If $p \in (1,n)$ and if $\sigma$ satisfies the conditions* (E0)–(E2)*, then the Dirichlet problem* (1.2) *has a weak solution $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ for every $v \in W^{-1,p'}(\Omega;\mathbb{R}^m)$, every $f$ satisfying* (F0)–(F1) *and every $g$ satisfying* (G0)–(G1)*.*

**Remark 1.6.** (a) For $p = n$, Theorem 1.5 holds if the conditions $b_2 \in L^\infty(\Omega)$ in (F1) and $b_5 \in L^\infty(\Omega)$ in (G1) are assumed. All the progress below remains valid with these adaptations.

(b) For $p > n$, Theorem 1.5 remains valid with $u \in C(\overline{\Omega};\mathbb{R}^m)$ if $b_2 \in L^1(\Omega)$ in (F1) and $b_5 \in L^1(\Omega)$ in (G1) hold.

(c) The strict bound $p-1$ for $\gamma$ and $\eta$ in the growth conditions (F1) and (G1) ensures the coercivity of the operator $F$ introduced in Section 4. However, the limit value $p-1$ is admissible in some particular cases (see Section 5).

(d) When $\sigma$ satisfies (c) or (d) in (E2), the function $f$ may even depend on the Jacobian matrix $Du$. See the subsequent articles of the authors.

The general structure of the proof of Theorem 1.5 follows [11] and [2].

## 2. A brief review on Young measures

Weak convergence is a basic tool of modern nonlinear analysis because it enjoys the same compactness properties that convergence in finite dimensional spaces does Evans90. Nonetheless, this notion does not behave as one would desire with respect to nonlinear functionals and operations. Young measures are a device to understand and to control these difficulties. The main theorem we will advocate to solve nonlinear PDEs systems is the following result due to Ball and proved in [3]:

**Theorem 2.1** (Ball)**.** *Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable, let $K \subset \mathbb{R}^m$ be closed, and let $u_j : \Omega \to \mathbb{R}^m$, $j \in \mathbb{N}$, be a sequence of Lebesgue measurable functions satisfying $u_j \to K$ in measure as $j \to \infty$, i.e. given any open neighborhood $U$ of $K$ in $\mathbb{R}^m$*
$$\lim_{j\to\infty} |\{x \in \Omega : u_j(x) \notin U\}| = 0.$$
*Then there exists a subsequence $(u_k)$ of $(u_j)$ and a family $(\nu_x)$, $x \in \Omega$, of positive measures on $\mathbb{R}^m$, depending measurably on $x$, such that*

(i) $\|\nu_x\|_{\text{meas}} \equiv \int_{\mathbb{R}^m} d\nu_x \leqslant 1$ *for a.e. $x \in \Omega$,*
(ii) $\operatorname{supp}\nu_x \subset K$ *for a.e. $x \in \Omega$, and*

(iii) $f(u_k) \overset{*}{\rightharpoonup} \langle \nu_x, f\rangle = \int_{\mathbb{R}^m} f(\lambda) d\nu_x(\lambda)$ *in* $L^\infty(\Omega)$ *for each continuous function* $f\colon \mathbb{R}^m \to \mathbb{R}$ *satisfying* $\lim_{|\lambda|\to\infty} f(\lambda) = 0$.

*Suppose further that* $\{u_k\}$ *satisfies the boundedness condition*

$$\forall R > 0\colon \lim_{L\to\infty} \sup_{k\in\mathbb{N}} |\{x \in \Omega \cap B_R\colon |u_k(x)| \geqslant L\}| = 0, \tag{2.1}$$

*where* $B_R = B_R(0)$. *Then*

$$\|\nu_x\|_{\text{meas}} = 1 \quad \text{for a.e. } x \in \Omega \tag{2.2}$$

*(i.e.* $\nu_x$ *is a probability measure), and there holds:*

*For any measurable* $A \subset \Omega$ *and any continuous function* $f\colon \mathbb{R}^m \to \mathbb{R}$ *such that* $\{f(u_k)\}$ *is sequentially weakly relatively compact in* $L^1(A)$ *we have* $f(u_k) \rightharpoonup \langle \nu_x, f\rangle$ *in* $L^1(A)$. (2.3)

Improved versions of this theorem exist: In [12, Theorem 1.2], it is shown that (2.1) is necessary for (2.2) and (2.3) to hold, and that in fact (2.1), (2.2) and (2.3) are equivalent.

In this article, we will adopt the following terminology:

**Convention 2.2.** Choosing $K = \mathbb{R}^m$, the assumptions of Ball's Theorem 2.1 are always fulfilled. Thus a family $(\nu_x)_{x\in\Omega}$ satisfying (i)–(iii) always exists. Moreover, once the subsequence $(u_k)$ of $(u_j)$ is fixed, $(\nu_x)_{x\in\Omega}$ obtained by this way is unique and is a sub-probability family on $\mathbb{R}^m$ by (i): A sub-probability family $(\tau_x)_{x\in\Omega}$ on $\mathbb{R}^m$ is a family of measures such that $\|\tau_x\|_{\text{meas}} \leqslant 1$ for a.e. $x \in \Omega$. Such a family $(\nu_x)_{x\in\Omega}$ is called a *Young measure* on $\Omega \times \mathbb{R}^m$. Thus, in this sense, each sequence generates a Young measure.

**Remark 2.3.** The notion of Young measure introduced in Convention 2.2 does not entirely coincide with the original definition of Young [16]. which is adopted in measure theory. For the link between these notions and a geometrical interpretation of Young measures, refer to [2] or [17].

Theorem 2.1 has useful applications, in particular in non-linear PDE theory. The following technical statements build the basic tools used in the next sections.

**Proposition 2.4.** *If* $|\Omega| < \infty$ *and* $\nu_x$ *is the Young measure (see Convention 2.2) generated by the (whole) sequence* $u_j$, *then there holds*

$$u_j \to u \text{ in measure if an only if } \nu_x = \delta_{u(x)} \text{ for a. e. } x \in \Omega.$$

For the proof, see [12, Proposition 1.3].

**Proposition 2.5.** *Let* $|\Omega| < \infty$. *If the sequences* $u_j\colon \Omega \to \mathbb{R}^m$ *and* $v_j\colon \Omega \to \mathbb{R}^d$ *generate the Young measures* $\delta_{u(x)}$ *and* $\nu_x$ *respectively, then* $(u_j, v_j)$ *generates the Young measure* $\delta_{u(x)} \otimes \nu_x$.

For the proof, see [12, Proposition 1.4]. This result also holds for sequences $\mu_j$, $\lambda_j$ of Young measures converging in the narrow topology to $\mu$ and $\lambda$ respectively; see [17]. However it is false in general if both $\mu$ and $\lambda$ are not Dirac measures. The third application is a Fatou-type lemma:

**Lemma 2.6.** *Let* $F : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{R}$ *be a Carathéodory function and* $u_k : \Omega \to \mathbb{R}^m$ *a sequence of measurable functions such that* $Du_k$ *generates the Young measure* $\nu_x$, *with* $\|\nu_x\|_{\text{meas}} = 1$ *for almost every* $x \in \Omega$. *Then*

$$\liminf_{k\to\infty} \int_\Omega F(x, u_k(x), Du_k(x))dx \geqslant \int_\Omega \int_{\mathbb{M}^{m\times n}} F(x, u, \xi) d\nu_x(\xi) dx$$

*provided that the negative part $F^-(x, u_k(x), Du_k(x))$ is equiintegrable.*

More general versions of this lemma may be found in [16], [17] and [14]. However, these assumptions allow a more elementary proof as in [12, Lemma 1.5].

## 3. A convergence result for $\sigma$

This section presents a general convergence result for functions $\sigma$ satisfying similar conditions as stated in Section 1. In fact, an elliptic div-curl inequality is the key ingredient to prove that one can pass to the limit in our quasilinear elliptic system. Since they are, in part, independent of the differential equation, we state them in a general form using only a set of hypotheses:

- (H1) The sequence $(u_k)$ is uniformly bounded in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ for some $p > 1$ and hence a subsequence converges weakly in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ to an element denoted by $u$.
- (H2) $\sigma\colon \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function.
- (H3) The sequence $\sigma_k(x) \equiv \sigma(x, u_k(x), Du_k(x))$ is uniformly bounded in the space $L^{p'}(\Omega;\mathbb{M}^{m\times n})$ and hence equiintegrable. The equiintegrability follows from the Hölder inequality.
- (H4) The sequence $(\sigma_k(x) : Du_k)^-$ is equiintegrable.
- (H5) There exists a sequence $(v_k)$ such that $v_k \to u$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ and $\int_\Omega \sigma_k(x) : (Du_k - Dv_k)dx \to 0$ as $k \to \infty$.

Note that the assumption (H1) ensures even a strong convergence in some Lebesgue spaces:

**Lemma 3.1.** *Let $p > 1$ and $(u_k)$ be a sequence which is uniformly bounded in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. Then there exists a subsequence of $(u_k)$ (for convenience not relabeled) and a function $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ such that*

$$u_k \rightharpoonup u \quad \text{in } W_0^{1,p}(\Omega;\mathbb{R}^m) \tag{3.1}$$

*and such that*

$$u_k \to u \quad \text{in measure on } \Omega \text{ and in } L^s(\Omega;\mathbb{R}^m) \tag{3.2}$$

*for all $s < p^*$.*

*Proof.* Since $(u_k)$ is bounded in $W_0^{1,p}(\Omega;\mathbb{R}^m)$, (3.1) follows directly from Eberlein-Smuljan Theorem [5]. Moreover, the Rellich-Kondrachov Theorem [1] implies that $(u_k)$ converges to an element $\tilde{u}$ in $L^s(\Omega;\mathbb{R}^m)$ for all $s < p^*$. Notice that in order to have the strong convergence simultaneously for all $s < p^*$, the usual diagonal sequence procedure applies. By unicity of the limit, $\tilde{u} = u$. Finally, the sequence converges in measure [9, Proposition 2.29] since $p > 1$. □

Now, under the conditions (H1)–(H5), we can prove the following div-curl inequality:

**Lemma 3.2** (div-curl inequality)**.** *Suppose* (H1)–(H5) *and assume (after passing to a suitable subsequence if necessary) that $(Du_k)$ generates the Young measure $\nu_x$. Then the following inequality holds:*

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : \lambda d\nu_x(\lambda)dx \leqslant \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda) : Dud\nu_x(\lambda)dx. \tag{3.3}$$

*Proof.* Let us consider the sequence

$$I_k \equiv \sigma(x, u_k, Du_k) : (Du_k - Du) = \sigma_k : Du_k - \sigma_k : Du.$$

By conditions (H3) and (H4), the negative part $I_k^-$ of $I_k$ is equiintegrable. Hence, we may use the Fatou-Lemma 2.6 which gives that

$$X \equiv \liminf_{k\to\infty} \int_\Omega I_k dx \geqslant \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, u, \lambda) : (\lambda - Du) d\nu_x(\lambda) dx.$$

It remains to prove that $X \leqslant 0$. For this, we note that by (H5) we have

$$\begin{aligned} X &= \liminf_{k\to\infty} \left( \int_\Omega \sigma_k : (Du_k - Dv_k) dx + \int_\Omega \sigma_k : (Dv_k - Du) dx \right) \\ &= \liminf_{k\to\infty} \int_\Omega \sigma_k : (Dv_k - Du) dx \leqslant \liminf_{k\to\infty} \underbrace{\|\sigma_k\|_{p'}}_{\leqslant C} \|v_k - u\|_{1,p} \ = \ 0, \end{aligned}$$

where we used the Hölder inequality and (H3). Thus the conclusion follows. $\square$

**Remark 3.3.** The naming "div-curl inequality" can be explained as follows. Suppose for a moment that div $\sigma(x, u_k, Du_k) = 0$ for all $k$ and that $\sigma(x, u_k, Du_k) : Du_k$ is equiintegrable. Then, the weak limit of $\sigma(x, u_k, Du_k) : Du_k$ in $L^1(\Omega)$ is given by $\int_{\mathbb{M}^{m\times n}} \sigma(x, u, \lambda) : \lambda d\nu_x(\lambda)$. On the other hand, by the usual div-curl lemma we conclude that $\int_\Omega \sigma(x, u_k, Du_k) : Du_k dx$ converges to $\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, u, \lambda) : Du d\nu_x(\lambda) dx$ and hence, the lemma would follow with equality.

The div-curl inequality will be the key ingredient to pass to the limit in the approximating equations. However, we need some additional information on the Young measure $\nu_x$ generated by the sequence of the gradients $(Du_k)$ to exploit (3.3). These properties are the following:

- (N1) $\nu_x$ is a probability measure for almost every $x \in \Omega$.
- (N2) $\nu_x$ is a homogeneous $W^{1,p}$ gradient Young measure for almost every $x \in \Omega$ in the sense that for $x \in \Omega$ fixed there exists a sequence $\tilde{u}(z)$ such that the Young measure $(\tilde{\nu}_z)_{z\in\Omega}$ generated by $D\tilde{u}(z)$ is homogeneous and equal to $\nu_x$: $\tilde{\nu}_z = \nu_x$ for almost every $z \in \Omega$.
- (N3) $\nu_x$ satisfies $\langle \nu_x, \mathrm{id} \rangle = Du(x)$ for almost every $x \in \Omega$.

The properties (N1)–(N3) follow in particular from the two estimates formulated in the next Lemma:

**Lemma 3.4.** *Let $\Omega$ be a bounded subset in $\mathbb{R}^n$ and $(u_k)_k$ a sequence in $W_0^{1,1}(\Omega; \mathbb{R}^m)$. Suppose that there exist $r > 0$, $p > 1$ and some constants $C$, $M$ and $L$ such that*

$$\sup_{k\in\mathbb{N}} \int_\Omega |u_k|^r dx \leqslant C$$

*and*

$$\sup_{k\in\mathbb{N}} \int_{|u_k|\leqslant R} |Du_k|^p dx \leqslant MR + L \quad \forall R > 0.$$

*Then the Young measure $\nu_x$ generated by (a subsequence of) $Du_k$ has finite $p$-th moment for almost every $x \in \Omega$ and satisfies* (N1)–(N3).

For the proof, see [8, Lemma 9]. In particular, (N1)–(N3) hold if the condition (H1) is fulfilled (actually if (H1) is satisfied, (N1)–(N3) can also be verified directly). In any case, the conditions (N1)–(N3) will be sufficient to pass to the limit as shown by the following convergence result for $\sigma$:

**Proposition 3.5.** *Suppose that* (H1)–(H5) *hold. Further assume that the Young measure*[1] $\nu_x$ *generated by the gradients* $Du_k$ *satisfies* (N1)–(N3) *and that one of the following conditions holds:*

(a) *The map* $F \mapsto \sigma(x,u,F)$ *is monotone and continuously differentiable for all* $(x,u) \in \Omega \times \mathbb{R}^m$.
(b) $\sigma(x,u,F) = \frac{\partial W}{\partial F}(x,u,F)$ *and* $F \mapsto W(x,u,F)$ *is a convex* $C^1$*-function for all* $(x,u) \in \Omega \times \mathbb{R}^m$.
(c) *The map* $F \mapsto \sigma(x,u,F)$ *is strictly monotone for all* $(x,u) \in \Omega \times \mathbb{R}^m$.
(d) *The map* $F \mapsto \sigma(x,u,F)$ *is strictly* $p$*-quasimonotone.*

*Then (after passage to a subsequence) the sequence* $\sigma_k$ *converges weakly in the space* $L^1(\Omega;\mathbb{M}^{m\times n})$ *as* $k \to \infty$ *and the weak limit* $\bar\sigma$ *is given by*

$$\bar\sigma(x) = \sigma(x,u(x),Du(x)).$$

*If* (b), (c) *or* (d) *holds, then*

$$\sigma(x,u_k(x),Du_k(x)) \to \sigma(x,u(x),Du(x)) \quad \text{in } L^1(\Omega;\mathbb{M}^{m\times n}).$$

*In cases* (c) *and* (d), *it follows in addition that (after extraction of a further subsequence if necessary)* $Du_k \to Du$ *in measure and almost everywhere in* $\Omega$.

Before we prove Proposition 3.5, we state a technical lemma which allows to localize the support of the Young measures $\nu_x$.

**Lemma 3.6.** *Suppose that* (H1)–(H5) *hold. Further assume that* $\nu_x$ *is the Young measure generated by the gradients* $Du_k$ *and satisfies* (N1)–(N3). *If the map* $F \mapsto \sigma(x,u,F)$ *is monotone for all* $(x,u) \in \Omega \times \mathbb{R}^m$, *then*

$$\operatorname{spt}\nu_x \subset \{\lambda \in \mathbb{M}^{m\times n} : (\sigma(x,u,\lambda) - \sigma(x,u,Du)) : (\lambda - Du) = 0\}. \tag{3.4}$$

*Proof.* By (N1) and (N3), we have (with $\bar\lambda = Du(x)$)

$$\begin{aligned}
&\int_{\mathbb{M}^{m\times n}} \sigma(x,u,\bar\lambda) : (\lambda - \bar\lambda) d\nu_x(\lambda) \\
&= \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\bar\lambda) : \lambda d\nu_x(\lambda) - \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\bar\lambda) : \bar\lambda d\nu_x(\lambda) \\
&= \sigma(x,u,\bar\lambda) : \underbrace{\int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)}_{=\bar\lambda} - \sigma(x,u,\bar\lambda) : \bar\lambda \underbrace{\int_{\mathbb{M}^{m\times n}} d\nu_x(\lambda)}_{=1} = 0.
\end{aligned}$$

By conditions (H1)–(H5), we have $\bar\lambda = Du(x)$ and we infer from inequality (3.3) in Lemma 3.2 that

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \big(\sigma(x,u,\lambda) - \sigma(x,u,\bar\lambda)\big) : \big(\lambda - \bar\lambda\big) d\nu_x(\lambda) dx \leqslant 0. \tag{3.5}$$

On the other hand, the integrand in (3.5) is non negative by monotonicity. It follows that the integrand must vanish almost everywhere with respect to the product measure $d\nu_x \otimes dx$. Hence, the conclusion follows. □

*Proof of Proposition 3.5.* We start with the easiest case:
**Case (c):** Since $\sigma$ is monotone by assumption, (3.4) holds by Lemma 3.6. By strict monotonicity, it follows from (3.4) that $\nu_x = \delta_{Du(x)}$ for almost all $x \in \Omega$, and hence $Du_k \to Du$ in measure for $k \to \infty$ by Proposition 2.4. Since we have already that

[1]The existence of $\nu_x$ is guaranteed by Ball's Theorem 2.1 (see Convention 2.2).

$u_k \to u$ in measure for $k \to \infty$ by (H1) and Lemma 3.1, we may infer that (after extraction of a suitable subsequence, if necessary [9, Theorem 2.30])

$$u_k \to u \quad \text{and} \quad Du_k \to Du \quad \text{almost everywhere in } \Omega \text{ for } k \to \infty.$$

From the continuity condition (H2), it follows that $\sigma(x, u_k, Du_k) \to \sigma(x, u, Du)$ almost everywhere in $\Omega$. Since, by assumption (H3), $\sigma_k(x)$ is equiintegrable, it follows from the Vitali convergence Theorem [9, Page 180] that

$$\sigma(x, u_k, Du_k) \to \sigma(x, u, Du) \quad \text{ in } L^1(\Omega; \mathbb{M}^{m\times n})$$

for $k \to \infty$, which proves the proposition in this case.
**Case (d):** Assume that $\nu_x$ is not a Dirac mass on a set $x \in M$ of positive Lebesgue measure $|M| > 0$. Then, by the strict $p$-quasimonotonicity of $\sigma(x, u, \cdot)$ and (N2), we have for a.e. $x \in M$ (with $\bar\lambda = \langle \nu_x, \mathrm{id}\rangle = Du(x)$ by (N3))

$$\begin{aligned}
&\int_{\mathbb{M}^{m\times n}} \sigma(x, u, \lambda) : \lambda d\nu_x(\lambda) \\
&> \underbrace{\int_{\mathbb{M}^{m\times n}} \sigma(x, u, \bar\lambda) : \lambda d\nu_x(\lambda)}_{=\sigma(x,u,\bar\lambda):\bar\lambda} - \underbrace{\int_{\mathbb{M}^{m\times n}} \sigma(x, u, \bar\lambda) : \bar\lambda d\nu_x(\lambda)}_{=\sigma(x,u,\bar\lambda):\bar\lambda\cdot 1} \\
&\quad + \int_{\mathbb{M}^{m\times n}} \sigma(x, u, \lambda) : \bar\lambda d\nu_x(\lambda) \\
&= \int_{\mathbb{M}^{m\times n}} \sigma(x, u, \lambda) : \bar\lambda d\nu_x(\lambda),
\end{aligned} \tag{3.6}$$

where we used (N1). We claim now that we obtain a contradiction. Indeed, by integrating (3.6) over $\Omega$ and using the div-curl inequality (3.3) in Lemma 3.2, we get

$$\begin{aligned}
\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, u, \lambda) : \lambda d\nu_x(\lambda) dx &> \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, u, \lambda) : \bar\lambda d\nu_x(\lambda) dx \\
&\geqslant \int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, u, \lambda) : \lambda d\nu_x(\lambda) dx
\end{aligned}$$

as desired. Hence, we have $\nu_x = \delta_{\bar\lambda} = \delta_{Du(x)}$ for almost every $x \in \Omega$. Thus, it follows again by Proposition 2.4 that $Du_k \to Du$ in measure for $k \to \infty$. The reminder of the proof in this case is exactly as in case (c).
**Case (b):** We start by showing that for almost all $x \in \Omega$, the support of $\nu_x$ is in the set where $W$ agrees with the supporting hyper-plane $L \equiv \{(\lambda, W(x, u, \bar\lambda) + \sigma(x, u, \bar\lambda)(\lambda - \bar\lambda))\}$ in $\bar\lambda = Du(x)$, i.e. we want to show that

$$\operatorname{spt} \nu_x \subset K_x = \{\lambda \in \mathbb{M}^{m\times n} : W(x, u, \lambda) = W(x, u, \bar\lambda) + \sigma(x, u, \bar\lambda) : (\lambda - \bar\lambda)\}.$$

Since $\sigma$ admits a potential, $\sigma$ is monotone and then (3.4) holds by Lemma 3.6. Thus, if $\lambda \in \operatorname{spt} \nu_x$ then by (3.4)

$$(1 - t)(\sigma(x, u, \bar\lambda) - \sigma(x, u, \lambda)) : (\bar\lambda - \lambda) = 0 \quad \text{for all } t \in [0, 1]. \tag{3.7}$$

On the other hand, by monotonicity, we have for $t \in [0, 1]$ that

$$0 \leqslant (1 - t)(\sigma(x, u, \bar\lambda + t(\lambda - \bar\lambda)) - \sigma(x, u, \lambda)) : (\bar\lambda - \lambda). \tag{3.8}$$

Subtracting (3.7) from (3.8), we get

$$0 \leqslant (1 - t)(\sigma(x, u, \bar\lambda + t(\lambda - \bar\lambda)) - \sigma(x, u, \bar\lambda)) : (\bar\lambda - \lambda) \tag{3.9}$$

for all $t \in [0,1]$. But by monotonicity, in (3.9) also the reverse inequality holds and we may conclude, that

$$(\sigma(x,u,\bar\lambda + t(\lambda-\bar\lambda)) - \sigma(x,u,\bar\lambda)) : (\lambda-\bar\lambda) = 0 \tag{3.10}$$

for all $t \in [0,1]$, whenever $\lambda \in \operatorname{spt}\nu_x$. Now, it follows from (3.10) that

$$\begin{aligned} W(x,u,\lambda) &= W(x,u,\bar\lambda) + \big(W(x,u,\lambda) - W(x,u,\bar\lambda)\big) \\ &= W(x,u,\bar\lambda) + \int_0^1 \sigma(x,u,\bar\lambda + t(\lambda-\bar\lambda)) : (\lambda-\bar\lambda) dt \\ &= W(x,u,\bar\lambda) + \sigma(x,u,\bar\lambda) : (\lambda-\bar\lambda) \end{aligned}$$

as claimed.

By the convexity of $W$ we have $W(x,u,\lambda) \geqslant W(x,u,\bar\lambda) + \sigma(x,u,\bar\lambda) : (\lambda-\bar\lambda)$ for all $\lambda \in \mathbb{M}^{m\times n}$ and thus $L$ is a supporting hyper-plane for all $\lambda \in K_x$. Since the mapping $\lambda \mapsto W(x,u,\lambda)$ is by assumption continuously differentiable we obtain

$$\sigma(x,u,\lambda) = \sigma(x,u,\bar\lambda) \quad \text{for all } \lambda \in K_x \supset \operatorname{spt}\nu_x \tag{3.11}$$

and thus

$$\bar\sigma(x) \equiv \int_{\mathbb{M}^{m\times n}} \sigma(x,u,\lambda)\, d\nu_x(\lambda) = \sigma(x,u,\bar\lambda)\,. \tag{3.12}$$

Now consider the Carathéodory function

$$\psi(x,u,p) = |\sigma(x,u,p) - \bar\sigma(x)|\,.$$

The sequence $\psi_k(x) = \psi(x,u_k(x),Du_k(x))$ is equiintegrable and thus by Ball's Theorem 2.1

$$\psi_k \rightharpoonup \bar\psi \quad \text{weakly in } L^1(\Omega)$$

and the weak limit $\bar\psi$ is given by

$$\begin{aligned} \bar\psi(x) &= \int_{\mathbb{R}^m\times\mathbb{M}^{m\times n}} |\sigma(x,\eta,\lambda) - \bar\sigma(x)|\, d\delta_{u(x)}(\eta) \otimes d\nu_x(\lambda) \\ &= \int_{\operatorname{spt}\nu_x} |\sigma(x,u(x),\lambda) - \bar\sigma(x)|\, d\nu_x(\lambda) = 0 \end{aligned}$$

by (3.11) and (3.12). Since $\psi_k \geqslant 0$ it follows that

$$\psi_k \to 0 \quad \text{strongly in } L^1(\Omega).$$

Thus the proof of the case (b) is finished.

**Case (a):** First we note that since $\sigma$ is monotone, (3.4) holds by Lemma 3.6. We claim that in this case for almost all $x \in \Omega$ the following identity holds for all $M \in \mathbb{M}^{m\times n}$ on the support of $\nu_x$:

$$\sigma(x,u,\lambda) : M = \sigma(x,u,\bar\lambda) : M + (\nabla_F\sigma(x,u,\bar\lambda)M) : (\bar\lambda - \lambda), \tag{3.13}$$

where $\nabla_F$ is the derivative with respect to the third variable of $\sigma$ and $\bar\lambda = Du(x)$. Indeed, by the monotonicity of $\sigma$ we have for all $t \in \mathbb{R}$

$$(\sigma(x,u,\lambda) - \sigma(x,u,\bar\lambda + tM)) : (\lambda - \bar\lambda - tM) \geqslant 0,$$

whence, by (3.4),

$$\begin{aligned} -\sigma(x,u,\lambda) : (tM) &\geqslant -\sigma(x,u,\bar\lambda) : (\lambda-\bar\lambda) + \sigma(x,u,\bar\lambda + tM) : (\lambda - \bar\lambda - tM) \\ &= t\big((\nabla_F\sigma(x,u,\bar\lambda)M)(\lambda-\bar\lambda) - \sigma(x,u,\bar\lambda) : M\big) + o(t). \end{aligned}$$

The claim follows from this inequality since the sign of $t$ is arbitrary. Since the sequence $\sigma_k(x)$ is equiintegrable by (H3) or by (C5), its weak $L^1$-limit $\bar\sigma$ is given by

$$\begin{aligned}\bar\sigma(x) &= \int_{\operatorname{spt}\nu_x} \sigma(x,u,\lambda)d\nu_x(\lambda)\\ &= \int_{\operatorname{spt}\nu_x} \sigma(x,u,\bar\lambda)d\nu_x(\lambda) + (\nabla_F\sigma(x,u,\bar\lambda))^t \underbrace{\int_{\operatorname{spt}\nu_x}(\bar\lambda-\lambda)d\nu_x(\lambda)}_{=\bar\lambda-\langle\nu_x,\mathrm{id}\rangle=0}\\ &= \sigma(x,u,\bar\lambda),\end{aligned}$$

where we used (3.13) in this calculation. This finishes the proof of the case (c) and hence of the proposition. □

**Remark 3.7.** In case (b), we remark, that the relation (3.12) already states that $\sigma(x,u,\bar\lambda)$ is the weak $L^1$-limit of $\sigma(x,u_k,Du_k)$, which is enough to pass to the limit in an equation which holds in the distributional sense. However, we wanted to point out that in this case, the convergence is even strong in $L^1(\Omega;\mathbb{M}^{m\times n})$.

## 4. Existence of a weak solution

To prove Theorem 1.5, we will apply a Galerkin scheme. First we recall that by the Poincaré and the Sobolev inequalities, there exists a constant $A \geqslant 1$ such that

$$\max(\|u\|_p, \|u\|_{p^*}) \leqslant A\|Du\|_p \quad \forall u \in W_0^{1,p}(\Omega;\mathbb{R}^m). \tag{4.1}$$

Note that we write $A$, in general without further comment, to point to the use of (4.1). This relation and the Hölder inequality are central to establish the required estimates to prove the desired results.

**Lemma 4.1.** *For arbitrary $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ and $v \in W^{-1,p'}(\Omega;\mathbb{R}^m)$, the functional $F(u): W_0^{1,p}(\Omega;\mathbb{R}^m) \to \mathbb{R}$ given by*

$$\begin{aligned}w \mapsto &\int_\Omega \sigma(x,u(x),Du(x)) : Dw(x)\,dx - \langle v,w\rangle - \int_\Omega f(x,u(x))\cdot w(x)\,dx\\ &+ \int_\Omega g(x,u(x)) : Dw(x)\,dx\end{aligned}$$

*is well defined, linear and bounded.*

*Proof.* On the one hand, the growth condition in (E1) allows us to estimate $I \equiv \int_\Omega \sigma(x,u,Du) : Dw dx$ for each $w \in W_0^{1,p}(\Omega;\mathbb{R}^m)$:

$$\begin{aligned}|I| &\leqslant \int_\Omega |\sigma(x,u,Du)||Dw|dx\\ &\leqslant \int_\Omega \lambda_1|Dw|dx + c_1\int_\Omega |u|^\beta|Dw|dx + c_1\int_\Omega |Du|^{p-1}|Dw|dx\\ &\leqslant \|Dw\|_p\big(\|\lambda_1\|_{p'} + c_1(A^{\frac{p^*}{p'}}\|Du\|_p^{p^*/p'} + \|Du\|_p^{p-1})\big),\end{aligned}$$

by the Hölder inequality and the bound for $\beta$. Next, the generalized Hölder inequality implies that

$$|\langle v,w\rangle| \leqslant \|v\|_{-1,p'}\|w\|_{1,p} \leqslant A\|v\|_{-1,p'}\|Dw\|_p.$$

On the other hand, if $II \equiv \int_\Omega f(x,u)\cdot w dx$, it follows from the growth condition (F1) (Without loss of generality, we may assume that $\gamma = p-1$). An application of the Hölder inequality to the three functions yields

$$\begin{aligned}|II| &\leqslant \int_\Omega |f(x,u)||w|dx \leqslant \int_\Omega b_1|w|dx + \int_\Omega b_2|u|^{p-1}|w|dx\\ &\leqslant \|b_1\|_{p'}\|w\|_p + \|b_2\|_{\frac{n}{p}}\|u\|_{p^*}^{p-1}\|w\|_{p^*}\\ &\leqslant \|Dw\|_p\Big(A\|b_1\|_{p'} + A^p\|b_2\|_{\frac{n}{p}}\|Du\|_p^{p-1}\Big).\end{aligned}$$

Finally, the growth condition (G1) (Without loss of generality, we may assume that $\eta = p-1$) allows us to estimate $III \equiv \int_\Omega g(x,u) : Dwdx$ for each $w \in W_0^{1,p}(\Omega;\mathbb{R}^m)$:

$$\begin{aligned}|III| &\leqslant \int_\Omega |g(x,u)||Dw|dx \leqslant \int_\Omega b_4|Dw|dx + \int_\Omega b_5|u|^{p-1}|Dw|dx\\ &\leqslant \|b_4\|_{p'}\|Dw\|_p + \|b_5\|_{\frac{n}{p-1}}\|u\|_{p^*}^{p-1}\|Dw\|_p\\ &\leqslant \|Dw\|_p\big(\|b_4\|_{p'} + A^{p-1}\|b_5\|_{\frac{n}{p-1}}\|Du\|_p^{p-1}\big).\end{aligned}$$

for each $w \in W_0^{1,p}(\Omega;\mathbb{R}^m)$. Since these four expressions are finite by our assumptions, $F(u)$ is well defined. Moreover, $F(u)$ is trivially linear and we have for all $w \in W_0^{1,p}(\Omega;\mathbb{R}^m)$

$$|\langle F(u), w\rangle| \leqslant |I| + |\langle v, w\rangle| + |II| + |III| \leqslant C\|Dw\|_p,$$

which implies that $F(u)$ is bounded. □

So we can define the operator

$$F : W_0^{1,p}(\Omega;\mathbb{R}^m) \to W^{-1,p'}(\Omega;\mathbb{R}^m), \quad u \mapsto F(u),$$

which satisfies the following property.

**Lemma 4.2.** *The restriction of $F$ to a finite dimensional linear subspace $V$ of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ is continuous.*

*Proof.* Let $r$ be the dimension of $V$ and $(\phi_i)_{i=1}^r$ a basis of $V$. Let $(u_j = a_j^i\phi_i)$ be a sequence in $V$ which converges to $u = a^i\phi_i$ in $V$ (with the standard summation convention). Then on the one hand the sequence $(a_j)$ converges to $a$ in $\mathbb{R}^r$ and so $u_j \to u$ and $Du_j \to Du$ almost everywhere and on the other hand $\|u_j\|_p$ and $\|Du_j\|_p$ are bounded by a constant $C$. Thus, it follows from the continuity conditions (E0), (F0)$^*$ and (G0) that $\sigma(x,u_j,Du_j) : Dw \to \sigma(x,u,Du) : Dw$, $f(x,u_j)\cdot w \to f(x,u)\cdot w$ and $g(x,u_j) : Dw \to g(x,u) : Dw$ almost everywhere. Moreover we infer from the growth conditions (E1), (F1) and (G1) that the sequences $(\sigma(x,u_j,Du_j) : Dw)$, $(f(x,u_j,Du_j)\cdot w)$ and $(g(x,u_j) : Dw)$ are equi-integrable. Indeed, if $\Omega' \subset \Omega$ is a measurable subset and $w \in W_0^{1,p}(\Omega;\mathbb{R}^m)$, then

$$\begin{aligned}&\int_{\Omega'} |\sigma(x,u_j,Du_j) : Dw|dx\\ &\leqslant \int_{\Omega'} \big(\lambda_1 + c_1(|u_j|^\beta + |Du_j|^{p-1})\big)|Dw|dx\\ &\leqslant \Big(\int_{\Omega'} |Dw|^p dx\Big)^{1/p}\big(\|\lambda_1\|_{p'} + c_1(A^{p^*/p'}\underbrace{\|Du_j\|_p^{p^*/p'}}_{\leqslant C} + \underbrace{\|Du_j\|_p^{p-1}}_{\leqslant C})\big)\end{aligned}$$

and (Without loss of generality, we may assume that $\gamma = p-1$),

$$\begin{aligned}\int_{\Omega'} |f(x,u_j)\cdot w|dx &\leqslant \int_{\Omega'} \big(b_1 + b_2|u_j|^{p-1}\big)|w|dx \\ &\leqslant A\Big(\int_{\Omega'} |Dw|^p dx\Big)^{\frac{1}{p}}\Big(\|b_1\|_{p'} + \big(A^{p-1}\|b_2\|_{\frac{n}{p}}\big)\underbrace{\|Du_j\|_p^{p-1}}_{\leqslant C}\Big)\end{aligned}$$

and (Without loss of generality, we may assume that $\eta = p-1$),

$$\begin{aligned}&\int_{\Omega'} |g(x,u_j) : Dw|dx \\ &\leqslant \Big(\int_{\Omega'} |Dw|^p dx\Big)^{\frac{1}{p}}\Big(\|b_4\|_{p'} + \big(A^{p-1}\|b_5\|_{\frac{n}{p-1}}\big)\underbrace{\|Du_j\|_p^{p-1}}_{\leqslant C}\Big)\end{aligned}$$

by the Hölder inequality (see the proof of Lemma 4.1). Applying the Vitali Theorem, it follows that for all $w \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ we have $\lim_{j\to\infty}\langle F(u_j), w\rangle = \langle F(u), w\rangle$ as desired. $\square$

**Remark 4.3.** Note that in all the progress in this subsection we used only the conditions $\gamma, \delta, \eta \leqslant p-1$. Thus Lemmas 4.1 and 4.2 are still valid as $\gamma = \delta = \eta = p-1$.

Now, the problem (1.2) is equivalent to find a solution $u \in W_0^{1,p}(\Omega;\mathbb{R}^m)$ such that

$$\langle F(u), w\rangle = 0 \quad \text{for all } w \in W_0^{1,p}(\Omega;\mathbb{R}^m). \tag{4.2}$$

In order to find such a solution we apply a Galerkin scheme. Let $V_1 \subset V_2 \subset \ldots \subset W_0^{1,p}(\Omega;\mathbb{R}^m)$ be a sequence of finite dimensional subspaces with the property that $\cup_{k\in\mathbb{N}} V_k$ is dense in $W_0^{1,p}(\Omega;\mathbb{R}^m)$[2]. Let us fix some $k$ and assume that $V_k$ has dimension $r$ and that $\phi_1,\ldots,\phi_r$ is a basis of $V_k$. Then we define the map

$$G : \mathbb{R}^r \to \mathbb{R}^r, \qquad \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^r \end{pmatrix} \mapsto \begin{pmatrix} \langle F(a^i\phi_i), \phi_1\rangle \\ \langle F(a^i\phi_i), \phi_2\rangle \\ \vdots \\ \langle F(a^i\phi_i), \phi_r\rangle \end{pmatrix}.$$

**Proposition 4.4.** *$G$ is continuous and*

$$G(a)\cdot a \to \infty \quad \textit{as } \|a\|_{\mathbb{R}^r} \to \infty.$$

*Proof.* Since $F$ restricted to $V_k$ is continuous by Lemma 4.2, $G$ is continuous. Let be $a \in \mathbb{R}^r$ and $u = a^i\phi_i \in V_k$. Then $G(a)\cdot a = \langle F(u), u\rangle$ and $\|a\|_{\mathbb{R}^r} \to \infty$ is equivalent to $\|u\|_{1,p} \to \infty$. Next, we note the following considerations. First the coercivity condition in (E1) and the Hölder inequality imply that

$$I \equiv \int_\Omega \sigma(x,u,Du) : Du dx \geqslant -\|\lambda_2\|_1 - A^\alpha\|\lambda_3\|_{(\frac{p}{\alpha})'}\|Du\|_p^\alpha + c_2\|Du\|_p^p.$$

Next the generalized Hölder inequality implies that

$$|II| \equiv |\langle v, u\rangle| \leqslant \|v\|_{-1,p'}\|u\|_{1,p} \leqslant A\|v\|_{-1,p'}\|Du\|_p.$$

[2]Such a sequence $(V_k)$ exists since $W_0^{1,p}(\Omega;\mathbb{R}^m)$ is separable.

Finally, it follows from the growth conditions (F1) and (G1) that (see the proof of Lemma 4.1)

$$III \equiv \int_\Omega f(x,u)\cdot u dx \leqslant A\|b_1\|_{p'}\|Du\|_p + A^{\gamma+1}\|b_2\|_{\frac{n}{p}}\|Du\|_p^{\gamma+1}$$

and

$$|IV| \equiv \Big|\int_\Omega g(x,u,Du) : Du dx\Big| \leqslant \|b_4\|_{p'}\|Du\|_p + A^\eta\|b_5\|_{\frac{n}{p-1}}\|Du\|_p^{\eta+1}.$$

From these estimations it follows that

$$\langle F(u),u\rangle = I - II - III + IV \to \infty \quad \text{as } \|u\|_{1,p}\to\infty,$$

since $p > \max(1,\alpha,\gamma+1,\delta+1,\eta+1)$ and $A, c_2 > 0$. □

The properties of $G$ allow us to construct our Galerkin approximations:

**Corollary 4.5.** *For all $k \in \mathbb{N}$ there exists $u_k \in V_k$ such that*

$$\langle F(u_k), w\rangle = 0 \quad \textit{for all } w \in V_k. \tag{4.3}$$

*Proof.* By Proposition 4.4 there exists $R > 0$ such that for all $a \in \partial B_R(0) \subset \mathbb{R}^r$ we have $G(a)\cdot a > 0$ and the usual topological argument [19, Proposition 2.8] gives that $G(x) = 0$ has a solution $x \in B_R(0)$. Hence, for all $k$ there exists $u_k \in V_k$ such that (4.3) holds. □

The Galerkin approximations satisfy the following bound:

**Proposition 4.6.** *The sequence of the Galerkin approximations constructed in* Corollary 4.5 *is uniformly bounded in $W_0^{1,p}(\Omega;\mathbb{R}^m)$, i.e. there exists a constant $R > 0$ such that*

$$\|u_k\|_{1,p} \leqslant R \quad \textit{for all } k \in \mathbb{N}. \tag{4.4}$$

*Proof.* As in the proof of Lemma 4.4 we see that $\langle F(u),u\rangle \to \infty$ as $\|u\|_{1,p}\to\infty$. Then it follows that there exists $R > 0$ with the property, that $\langle F(u),u\rangle > 1$ whenever $\|u\|_{1,p} > R$. Thus, for the sequence of Galerkin approximations $u_k \in V_k$ which satisfy $\langle F(u_k),u_k\rangle = 0$ by (4.3), we have the uniform bound (4.4). □

Now, we are able to pass to the limit and so to prove Theorems 1.5. First, in order to apply Proposition 3.5, we verify that, under our assumptions, the conditions (H1)–(H5) and (N1)–(N3) hold for the Galerkin approximations solutions $u_k$ constructed before.

(H1) holds by Proposition 4.6. Moreover, it follows then by Lemma 3.4 that (N1)–(N3) hold.

The condition (H2) is equivalent to (E0). To obtain (H3), we observe that by the growth condition in (E1)

$$\int_\Omega |\sigma(x,u_k,Du_k)|^{p'}dx \leqslant C\Big(\int_\Omega (|\lambda_1(x)|^{p'} + |u_k|^{p^*} + |Du_k|^p)dx\Big),$$

which is uniformly bounded in $k$ by (4.4) since $\|u_k\|_{p^*} \leqslant A\|Du_k\|_p$ by (4.1).

Next, to verify (H4), we fix an arbitrary measurable subset $\Omega' \subset \Omega$. Then, on the one hand, the growth condition in (E1) implies that

$$\begin{aligned}
&\int_{\Omega'} |\min(\sigma(x,u_k,Du_k):Du_k,0)|dx \\
&\leqslant \int_{\Omega'} |\lambda_2|dx + \int_{\Omega'} |\lambda_3||u_k|^\alpha dx \\
&\leqslant \int_{\Omega'} |\lambda_2|dx + \Big[\int_{\Omega'} |\lambda_3|^{(\frac{p}{\alpha})'}dx\Big]^{1/(\frac{p}{\alpha})'} \underbrace{\Big[\int_{\Omega'} |u_k|^p dx\Big]^{\alpha/p}}_{\leqslant R^\alpha}
\end{aligned}$$

by the Hölder inequality and (4.4). Since a finite set of integrable functions is equi-integrable, the equi-integrability of $(\sigma_k : Du_k)^-$ follows.

Finally, we want to prove (H5): According to Mazur's Theorem (see, e.g., [18, Theorem 2, page 120]) there exists a sequence $v_k$ in $W_0^{1,p}(\Omega)$ where each $v_k$ is a convex linear combination of $\{u_1,\dots,u_k\}$ such that $v_k \to u$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$. I.e.

$$v_k \text{ belongs to the same space } V_k \text{ as } u_k \text{ and } v_k \to u \text{ in } W_0^{1,p}(\Omega;\mathbb{R}^m). \tag{4.5}$$

This allows us in particular, to use $u_k - v_k$ as a test function in (4.3). We have

$$\begin{aligned}
&\int_\Omega \sigma(x,u_k,Du_k):(Du_k - Dv_k)dx \\
&= \langle v, u_k - v_k\rangle + \int_\Omega f(x,u_k)\cdot(v_k-u_k)dx - \int_\Omega g(x,u_k):(Dv_k - Du_k)dx.
\end{aligned} \tag{4.6}$$

The first term on the right in (4.6) converges to zero since

$$u_k - v_k \rightharpoonup 0 \quad \text{in } W_0^{1,p}(\Omega;\mathbb{R}^m) \tag{4.7}$$

by the choice of $v_k$, (H1) and Lemma 3.1. Now, for the second term $II_k \equiv \int_\Omega f(x,u_k)\cdot(v_k-u_k)dx$ in (4.6) it follows from the growth condition (F1) and the Hölder inequality that

$$|II_k| \leqslant \|b_1\|_{p'}\|v_k-u_k\|_p + \|b_2\|_{\frac{n}{p}}\|u_k\|_{p^*}^\gamma \|v_k-u_k\|_{\frac{p^*}{p-\gamma}}.$$

By (4.1) and (4.4), $\|u_k\|_{p^*}$ is bounded. Moreover, by the construction of $v_k$, (H1) and Lemma 3.1 we have

$$\|u_k - v_k\|_s \leqslant \|u_k - u\|_s + \|u - v_k\|_s \to 0$$

for all $s < p^*$. Since it follows from $\gamma < p-1$ that $\frac{p^*}{p-\gamma} < p^*$, we infer that the second term in (4.6) vanishes as $k \to \infty$. Finally, for the last term $III_k \equiv \int_\Omega g(x,u_k) : D(v_k-u_k)dx$ in (4.6) we note that $g(x,u_k) \to g(x,u)$ strongly in $L^{p'}(\Omega;\mathbb{M}^{m\times n})$ by (G0), (G1), (H1) and (3.2) in Lemma 3.1. Indeed we may assume by (H1) that $u_k \to u$ almost everywhere (since in the sequel we consider only a subsequence of $u_k$, not relabeled for convenience, which converges almost everywhere to $u$). Since by (G1) $|g(x,u_k)|^{p'}$ is bounded by an integrable function

$$|g(x,u_k)|^{p'} \leqslant C(|b_4|^{p'} + |b_5|^{p'}|u_k|^p) \leqslant C(|b_4|^{p'} + |b_5|^{p'}(1+|u|^p)) \in L^1(\Omega),$$

the claim follows from (G0) and the Dominated Convergence Theorem [9, Theorem 2.24]. We infer then that

$$|III_k| \leqslant \|g(u_k) - g(u)\|_{p'} \underbrace{\|Dv_k - Du_k\|_p}_{\leqslant C} + \Big| \int_\Omega g(x,u) : D(v_k - u_k) dx \Big| \to 0$$

as $k \to \infty$ by (4.7). Thus the last term in (4.6) disappears also as $k \to \infty$ and (H5) is fulfilled.

So, (H1)–(H5) and (N1)–(N3) hold and we may infer from Proposition 3.5 that

$$\lim_{n\to\infty} \int_\Omega \sigma(x, u_k, Du_k) : Dw(x) dx = \int_\Omega \sigma(x, u, Du) : Dw(x) dx \quad \forall w \in \bigcup_{k=1}^\infty V_k.$$

Moreover, since $u_k \to u$ in measure for $k \to \infty$, we may infer that, after extraction of a suitable subsequence, if necessary, [9, Theorem 2.30]

$$u_k \to u \quad \text{almost everywhere for } k \to \infty.$$

Thus, for arbitrary $w \in W_0^{1,p}(\Omega; \mathbb{R}^m)$, it follows from the continuity conditions (F0) and (G0) that $f(x, u_k) \cdot w(x) \to f(x, u) \cdot w(x)$ and $g(x, u_k) : Dw(x) \to g(x, u) : Dw(x)$ almost everywhere. Since, by the growth conditions (F1) and (G1) and the uniform bound (4.4), $f(x, u_k) \cdot w(x)$ and $g(x, u_k) : Dw(x)$ are equiintegrable (see the proof of Lemma 4.2), it follows that $f(x, u_k) \cdot w(x) \to f(x, u) \cdot w(x)$ and $g(x, u_k) : Dw(x) \to g(x, u) : Dw(x)$ in $L^1(\Omega)$ by the Vitali convergence Theorem. This implies that

$$\lim_{k\to\infty} \int_\Omega f(x, u_k(x)) \cdot w(x) dx = \int_\Omega f(x, u(x)) \cdot w(x) dx \quad \forall w \in \bigcup_{k=1}^\infty V_k$$

and

$$\lim_{k\to\infty} \int_\Omega g(x, u_k(x)) : Dw(x) dx = \int_\Omega g(x, u(x)) : Dw(x) dx \quad \forall w \in \bigcup_{k=1}^\infty V_k.$$

Since $\bigcup_{k=1}^\infty V_k$ is dense in $W_0^{1,p}(\Omega; \mathbb{R}^m)$, $u$ is then a weak solution of (1.2), as desired.

## 5. Improved result in a particular case

In this section we consider again the elliptic system (1.2). In the previous section, we optimized the various parameters in the different assumptions on $f$ and $g$. However, in some particular case, our main assumptions on $f$ and $g$ may be weakened. We focus in particular our attention on the case where the limit bound $p - 1$ for $\gamma$ and $\eta$ is admissible.

We consider only the case $p \in (1, n)$. For the cases $p = n$ and $p > n$, refer to Remarks 1.6.

As seen in the proofs of Theorem 1.5, the strict bound $p - 1$ for $\gamma$ and $\eta$ in the growth conditions (F1) and (G1) ensures the coercivity of the operator $F$ introduced in the previous section. However, when the norms of $b_2$ and $b_5$ are small enough, the limit bound $p - 1$ in (F1) and (G1) is allowed for $\gamma$, $\delta$, and $\eta$. More exactly if

(C) $c_2 > \chi(\gamma) A^{\gamma+1} \|b_2\|_{\frac{n}{p}} + \chi(\eta) A^\eta \|b_5\|_{\frac{n}{p-1}}$

where

$$\chi(\xi) = \begin{cases} 1 & \text{if } \xi = p - 1 \\ 0 & \text{if } 0 < \xi < p - 1 \end{cases} \tag{5.1}$$

and $A$ is given by (4.1), the factor multiplying $\|Du\|_p^p$ remains strictly positive. So, the argumentation is still valid in this special case and Propositions 4.4 and 4.6 hold. Note that the condition (C) is in particular fulfilled when the measure of $\Omega$ is small enough.

However, we cannot infer that Theorem 1.5 holds when $\gamma, \eta \leqslant p-1$ even though the condition (C) is satisfied. Indeed, we use the strict bound for $\gamma$ to verify that the condition (H5) is fulfilled and then to apply Proposition 3.5. So, in order to allow the bound $p-1$ for the different parameters, we have to set stronger assumptions on the coefficient $b_2$:

**Theorem 5.1.** *Let be* $p \in (1, n)$*,* $\sigma$ *satisfying the conditions* (E0)–(E2)*,* $v \in W^{-1,p'}(\Omega; \mathbb{R}^m)$*,* $f$ *satisfying* (F0)–($\overline{\text{F1}}$) *and* $g$ *satisfying* (G0)–(G1) *with* $0 < \eta \leqslant p-1$ *where*

($\overline{\text{F1}}$) *(Growth) There exist* $0 < \gamma \leqslant p-1$, $q > \frac{n}{p}$, $b_1 \in L^{p'}(\Omega)$ *and* $b_2 \in L^q(\Omega)$ *such that*
$$|f(x,u)| \leqslant b_1(x) + b_2(x)|u|^\gamma.$$

*If the condition* (C) *is fulfilled, then the Dirichlet problem* (1.2) *has a weak solution* $u \in W_0^{1,p}(\Omega; \mathbb{R}^m)$*.*

*Proof.* Note first that the conclusions of Lemmas 4.1 and 4.2 hold in view of Remark 4.3. Moreover Propositions 4.4 and 4.6 hold by (C). Then, as before, we verify easily that the conditions (H1)–(H4) and (N1)–(N3) are fulfilled. To prove that (H5) holds, we choose a sequence $v_k$ such that (4.5) holds. This allows us in particular, to use $u_k - v_k$ as a test function in (4.3). We have
$$\begin{aligned} &\int_\Omega \sigma(x, u_k, Du_k) : (Du_k - Dv_k)dx \\ &= \langle v, u_k - v_k\rangle + \int_\Omega f(x,u_k)\cdot(v_k - u_k)dx - \int_\Omega g(x,u_k) : (Dv_k - Du_k)dx. \end{aligned} \tag{5.2}$$
The first and the third terms in (5.2) converge to zero as seen before. For the second term $II_k \equiv \int_\Omega f(x,u_k)\cdot(v_k-u_k)dx$ in (5.2) it follows from the growth condition ($\overline{\text{F1}}$) and the Hölder inequality that
$$|II_k| \leqslant \|b_1\|_{p'}\|v_k - u_k\|_p + \|b_2\|_q\|u_k\|_{p^*}^{p-1}\|v_k - u_k\|_{s(q)},$$
where
$$s(q) \equiv \frac{q'np}{np - q'(n-p)(p-1)}.$$
By (4.4), $\|u_k\|_{p^*}$ and $\|Du_k\|_p$ are bounded. Moreover, by the construction of $v_k$, (H1) and Lemma 3.1 we have
$$\|u_k - v_k\|_s \leqslant \|u_k - u\|_s + \|u - v_k\|_s \to 0$$
for all $s < p^*$. Since it follows from $q > n/p$ that $s(q) < p^*$, we infer that the second term in (5.2) vanishes as $k \to \infty$ and thus (H5) is fulfilled. Now the proof ends as in the previous section. □

## References

[1] R. A. Adams. *Sobolev spaces.* Academic Press, New York, 1975.

[2] F. Augsburger. Young measures and quasilinear systems in divergence form with weak monotonicity. PhD Thesis, Université de Fribourg, 2004.

[3] J.M. Ball. A version of the fundamental theorem for Young measures. In *Partial differential equations and continuum models of phase transitions: Proceedings of an NSF-CNRS joint seminar*, 1989.

[4] H. Brezis. *Opérateurs maximaux monotones*. North-Holland Publishing Company, Amsterdam, 1973.

[5] H. Brezis. *Analyse fonctionnelle*. Masson, Paris, 1983.

[6] F. E. Browder. Existence theorems for nonlinear partial differential equations. *Global Analysis (Proc. Sympos. Pure Math., Vol XVI, Part 2, Berkeley, 1968)*, pages 1–60, 1968.

[7] J. Chabrowski and K.-W. Zhang. Quasi-monotonicity and perturbated systems with critical growth. *Indiana Univ. Math. J.*, 41(2):483–504, 1992.

[8] G. Dolzmann, N. Hungerbühler, and S. Müller. Non-linear elliptic systems with measure-valued right hand side. *Matematische Zeitschrift*, 226:545–574, 1997.

[9] G. B. Folland. *Real analysis: Modern technics and their applications*. Wiley Interscience Publication, New York, 1973.

[10] M. Fuchs. Regularity theorems for nonlinear systems of partial differential equations under natural ellipticity conditions. *Analysis*, 7(1):83–93, 1987.

[11] N. Hungerbühler. Quasilinear elliptic systems in divergence form with weak monotonicity. *New York J. Math.*, 5:83–90, 1999.

[12] N. Hungerbühler. Young measures and nonlinear PDEs. Birmingham, 1999.

[13] D. Kinderlehrer and P. Pedregal. Young measures generated by sequences in Sobolev spaces. *J. Geom. Anal.*, 4(1):59–90, 1994.

[14] J. Kristensen. Lower semicontinuity in spaces of weakly differentiable functions. *Math. Annalen*, 313(4):653–710, 1999.

[15] J.L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Etudes mathématiques. Dunod, Paris, 1969.

[16] M. Valadier. Young measures. In *Methods of nonconvex analysis*, volume 1146 of *Lectures Notes in Math.*, pages 152–188. Springer, 1990.

[17] M Valadier. A course on Young measures. *Worshop on Measure Theory and Real Analysis (Italian). Rend. Istit. Mat. Univ. Triestre*, 26:suppl. 349–394 (1995), 1994.

[18] K. Yosida. *Functional analysis*. Springer, Berlin, 1980.

[19] E. Zeidler. *Nonlinear functional analysis and its application*, volume I. Springer, 1986.

[20] E. Zeidler. *Nonlinear functional analysis and its application*, volume II/B. Springer, 1990.

[21] E. Zeidler. *Nonlinear functional analysis and its application*, volume II/A. Springer, 1990.



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